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THE LAW OF THE MEAN AND THE LIMITS $\frac{0}{0}, \frac{\infty}{\infty}$.

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1. The very terms "indeterminate form" and "true value" indicate the manner in which the ratio $\dfrac{f(x)}{F(x)}$ was regarded by the older mathematicians, when for a particular value of x, $x = a$, $f(x)$ and $F(x)$ both vanish or both become infinite. These terms imply that the function

$$\psi(x) = \frac{f(x)}{F(x)} \tag{1}$$

really has a value when $x = a$, but that the *form* in which the function is given us is unsuited to the determination of this value; and so it was an easy step for the ascertainment of this "true value" to allow x to approach a as its limit and then seek the limit approached by $\psi(x)$. In fact, however, the above formula, while defining the function $\psi(x)$ in general, does *not* define it for the value $x = a$, and it would be possible, so far as logical reasons go, to define $\psi(a)$ as any quantity we may choose to select. When we agree, therefore, to assign to $\psi(x)$ for the value $x = a$ the limit that $\psi(x)$ approaches when x approaches a, we are adding, *arbitrarily*, to the definition of $\psi(x)$ contained in (1) a definition neither contained nor implied in (1). The reason for this definition is, of course, that in the cases that most frequently arise in practice $\psi(x)$ is a continuous function for values of x different from a, and if $\psi(a)$ is defined as above, then $\psi(x)$ will be continuous for the value $x = a$ too.

2. The erroneous view above referred to is in itself a matter of small importance; but it tends to obscure an essential part in a question of considerable importance. The problem: *to determine*

$$\lim_{x=a} \frac{f(x)}{F(x)}$$

when either (1) $f(a) = 0$, $F(a) = 0$ *or* (2) $f(a) = \infty$, $F(a) = \infty$ is a twofold problem, consisting (I) in ascertaining whether the variable $f(x)/F(x)$ actually converges toward a limit and (II) in case a limit exists, in finding what this limit is. This problem is solved for the cases that arise in ordinary practice by the aid of the following theorem, a simple and rigorous proof of which forms the subject of the second part of this paper.

THEOREM: *If* (1) $f(a) = 0$, $F(a) = 0$ *or* (2) $f(a) = \infty$, $F(a) = \infty$, *and if*

$$\frac{f'(x)}{F'(x)}$$

converges toward a limit when x approaches a, then (I) $\frac{f(x)}{F(x)}$ *also converges toward a limit and* (II) *these limits are equal:*

$$\lim_{x=a} \frac{f(x)}{F(x)} = \lim_{x=a} \frac{f'(x)}{F'(x)}.$$

Here a may denote either a fixed quantity or $+\infty$, $-\infty$; and the cases that $\lim_{x=a} \frac{f'(x)}{F'(x)} = +\infty, -\infty$ *are also included.*

Remark. If $f'(a)$, $F'(a)$ are both 0 or both infinite, the theorem can be applied anew to the ratio $f'(x)/F'(x)$. Hence, in this case,

$$\lim_{x=a} \frac{f(x)}{F(x)} = \lim_{x=a} \frac{f''(x)}{F''(x)}; \quad \text{etc.}$$

3. Let us first consider the treatment ordinarily given in the works on calculus for determining $\lim_{x=a} \frac{f(x)}{F(x)}$, when $f(a) = 0$, $F(a) = 0$; or as I will say more briefly, the limit $\frac{0}{0}$. x is written as $a + h$; then

$$\frac{f(x)}{F(x)} = \frac{\dfrac{f(a+h) - f(a)}{h}}{\dfrac{F(a+h) - F(a)}{h}},$$

whence it is inferred that

$$\lim_{x=a} \frac{f(x)}{F(x)} = \frac{f'(a)}{F'(a)};$$

i. e. (I) that $f(x)/F(x)$ approaches a limit and (II) that this limit is $f'(a)/F'(a)$.*

* In his *Analyse des Infiniment Petits*, Paris, 1696, l'Hospital gives the theorem and proof in the following form (p. 145; the figure is an exact copy):

"*Soit une ligne courbe* AMD (AP = x, PM = y, AB = a) *telle que la valeur de l'appliquée* y *soit exprimée par une fraction, dont le numérateur et le dénominateur deviennent chacun zero lorsque*

This is correct provided $f'(a)/F'(a)$ has a meaning, i. e. is not of the form $0/0$ or ∞/∞. But suppose that, as is frequently the case, $f'(a) = 0$, $F'(a) = 0$. Then we are told to differentiate again, forming $f''(a)/F''(a)$; etc. For $f'(a)/F'(a)$ really has a meaning, according to the fallacious view above cited, only the "true value" eludes us in the unfortunate form that $f'(a)/F'(a)$ has taken on. Now here is a positive error, and it consists in the assumption that the "true value of $f'(a)/F'(a)$" obtained as the limit

$$\lim_{h=0} \frac{\dfrac{f(a+h)-f(a)}{h}}{\dfrac{F(a+h)-F(a)}{h}}$$

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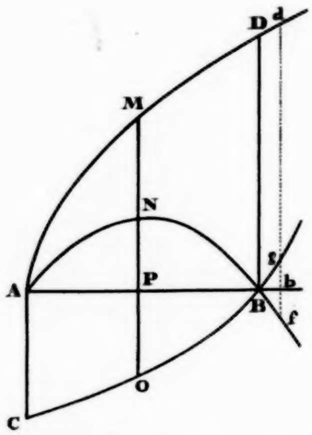
x = a, *c'est-à-dire lorsque le point* P *tombe sur le point donné* B. *On demande quelle doit être alors la valeur de l'appliquée* BD.



"Soient entenduës deux lignes courbes ANB, COB, qui ayent pour axe commun la ligne AB, et qui soient telles que l'appliquée PN exprime le numérateur, et l'appliquée PO le dénominateur de la fraction générale qui convient à toutes les PM: de sorte que $PM = \frac{AB \times PN}{PO}$. Il est clair que ces deux courbes se rencontreront au point B: puisque par la supposition PN et PO deviennent chacune zero lorsque le point P tombe en B. Cela posé, si l'on imagine une appliquée bd infiniment proche de BD, et qui rencontre les lignes courbes ANB, COB aux points f, g; l'on aura $bd = \frac{AB \times bf}{bg}$, laquelle ne diffère pas de BD. Il n'est donc question que de trouver le rapport de bg à bf. Or il est visible que la coupée AP devenant AB, les appliquées PN, PO deviennent nulles; et que AP devenant Ab, elles deviennent bf, bg. D'où il suit que ces appliquées, elles-mêmes bf, bg, sont la différence des appliquécs en B et b par rapport aux courbes ANB, COB; et partant que si l'on prend la différence du numérateur, et qu'on la divise par la différence du dénominateur, aprés avoir fait $x = a = Ab$ ou AB, l'on aura la valeur cherchée de l'appliquée bd ou BD. Ce qu'il falloit trouver."

Then follow the examples $y = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}}$, $y = \frac{a^2 - ax}{a - \sqrt{ax}}$ with their solutions.

is the same quantity as the "true value of $f'(a)/F'(a)$" obtained as the limit

$$\lim_{x=a} \frac{f'(x)}{F'(x)}.$$

There is no reason in what has preceded why these two limits should stand in any relation to each other. For those readers who are familiar with the subject of double limits* the fallacy can be thrown into still more striking form as follows. Write

$$\frac{\dfrac{f(a+h)-f(a)}{h}}{\dfrac{F(a+h)-F(a)}{h}} = \lim_{\xi=a} \frac{\dfrac{f(\xi+h)-f(\xi)}{h}}{\dfrac{F(\xi+h)-F(\xi)}{h}}$$

and

$$\frac{f'(\xi)}{F'(\xi)} = \lim_{h=0} \frac{\dfrac{f(\xi+h)-f(\xi)}{h}}{\dfrac{F(\xi+h)-F(\xi)}{h}}$$

Then the "true value of $f'(a)/F'(a)$" obtained in the first way is

$$\lim_{x=a} \frac{f(x)}{F(x)} \quad \text{or} \quad \lim_{h=0} \left\{ \lim_{\xi=a} \frac{\dfrac{f(\xi+h)-f(\xi)}{h}}{\dfrac{F(\xi+h)-F(\xi)}{h}} \right\};$$

while the "true value of $f'(a)/F'(a)$" obtained in this second way is

$$\lim_{\xi=a} \frac{f'(\xi)}{F'(\xi)} \quad \text{or} \quad \lim_{\xi=a} \left\{ \lim_{h=0} \frac{\dfrac{f(\xi+h)-f(\xi)}{h}}{\dfrac{F(\xi+h)-F(\xi)}{h}} \right\},$$

and it remains to be proven that these two double limits are equal to each other.

Thus this method for establishing the rule for evaluating the limit 0/0 is seen to be without foundation in a large class of cases, and the correctness of the rule itself in these cases is thereby called into question.

A second method of establishing the rule consists in developing numerator and denominator into power series according to powers of $x - a$. But aside

* For an elementary treatment of several questions in double limits cf. the writer's article: *A Geometrical Method for the Treatment of Uniform Convergence and Certain Double Limits* in the November number of the *Bulletin of the Amer. Math. Soc.*, 2nd ser., vol. III, 1896, pp. 59–86.

from the fact that the elaborate methods of infinite series are thus called into play to prove an elementary theorem, the rule is not even then established in the generality required in practice. Thus both methods would fail to prove the rule for the simple example

$$\lim_{x=1} \frac{x^{4/3} - \frac{4}{3}x + \frac{1}{3}}{(x-1)^{4/3}}.$$

4. The term "true value" (*véritable valeur* or *vraie valeur*) was accepted by Cauchy and there is nothing in his presentation of the subject of "indeterminate forms" in his two works on the differential calculus* to guard his readers against the wrong view above cited. In his *Analyse algébrique* of 1821 he sets the subject in its right light without however emphasizing the fact that we have to do here with two distinct and independent definitions of the function $\phi(x)$.†

In the *Cours d'Analyse* of 1823 he devotes half a page (p. 24) to the determination of the limit 0/0, giving l'Hospital's method in purely arithmetic form without commenting on the cases where it ceases to apply. But in the *Calcul différentiel* of 1826 he devised, in order to meet these cases, a method which in motif, simplicity and rigor leaves nothing to be desired—the method set forth in this paper. One oversight is noteworthy. Although Cauchy had in the theorem

$$\frac{f(x)}{F(x)} = \frac{f'(X)}{F'(X)}$$

all the material for the proof of the *existence of a limit* for $f(x)/F(x)$ under the hypothesis that $f'(x)/F'(x)$ approaches a limit, it did not occur to him to raise and dispose of this important question. On such phenomena we can measure the advance in analysis since Cauchy's time.

5. The rule for evaluating the limit ∞/∞ (Case (2) of the Theorem of § 2) together with the proof ordinarily given (cf. for example Williamson's Differential Calculus, Ch. IV) appears in Cauchy's *Calcul différentiel* (pp. 41, 42). Here again only the second half of the theorem is proven, it being *assumed* that $f(x)/F(x)$ approaches a limit. In his *Analyse algébrique* Cauchy had proved two lemmas regarding the limit of $f(x)/x$ and $[f(x)]^{\frac{1}{x}}$ when $f(\infty) = \infty$ and by transformation to these forms had disposed of the most common

* *Cours d'Analyse*, the exact title being *Résumé des leçons sur le calcul infinitésimal*, t. I., Paris, 1823; *Leçons sur le calcul différentiel*, Paris, 1829.

† The heading of the paragraph (p. 45) is: *Valeurs singulières des Fonctions dans quelques cas particuliers* and the term *vraie valeur* nowhere appears in this section. It is to be regretted that mathematicians did not generally (De Morgan is an exception) adopt the term *singular value* in place of *true value* and the writer would suggest that it is not too late to make the change.

limits of these types; such as $\log x/x$, a^x/x, $x^{\frac{1}{x}}$, when $x = \infty$, etc. And in the *Cours d'Analyse* of 1823 there is nothing new on this subject. Whether the rule was known at that earlier time, but the proof was too loose to satisfy him, or whether he was himself the discoverer of the rule,—it is not likely that he would have overlooked it, had it been discovered earlier,—the writer cannot say.

Du Bois-Reymond used the Law of the Mean to prove Case (2) of the Theorem of § 2 for the special case that $F(x) = x$. Stolz gave the proof in the general case,* and his proof is the one now usually given.† It is reproduced in substance, but given in more elementary form, in § 12 of this paper.

The Law of the Mean.

6. When mathematicians began, early in this century, to turn their attention to obtaining a rigorous foundation for analysis to rest upon, they found that a theorem that is usually referred to as the Law of the Mean is of fundamental importance for the Differential Calculus.‡ Hitherto the writers of text-books have treated this law as if it belonged to the higher regions of analysis; while in fact, if rigor and simplicity are to characterize the treatment, this law and the methods based on it are indispensable from the beginning.

The proof of the law usually given §, though simple and rigorous, is lacking in one important respect, for it does not emphasize sufficiently strongly the thought that underlies the proof. For that reason the writer thinks it desirable to reproduce the proof thus modified.

THEOREM I. *Let $\varphi(x)$ be a single valued continuous function of x in the interval $a \leq x \leq b$ and let it have a derivative‖ for all values of x within*

* *Math. Ann.* vol. 14, 1879; also Stolz, *Differential- und Integralrechnung*, p. 77, where the literature is cited.

† Peano, *Calcolo differenziale*, Turin, 1884; Tannery, *Introduction à la théorie des fonctions d'une variable*, Paris, 1886; as well as Stolz, already cited.

‡ The name Law of the Mean is also applied to certain well known theorems of the Integral Calculus; but the context usually shows which one of these laws is meant.

§ This proof is ascribed to O. Bonnet. Cf. Serret, *Cours d'Analyse*.

‖ Confusion sometimes arises from failure to notice the definition of a derivative. The function $\varphi(x)$ is said to have a derivative in the point x_0 if the variable

$$\frac{\varphi(x_0 + \Delta x) - \varphi(x_0)}{\Delta x}$$

converges toward a fixed limit or becomes positively or negatively infinite, when Δx converges continuously toward 0, *passing both through positive and through negative values.* Thus if for $x = x_0$ the curve $y = \varphi(x)$ has a cusp with tangent parallel to the y-axis, $\varphi(x)$ has no derivative for $x = x_0$.

this interval: $a < x < b$. *Then for at least one mean value* X *of* x

$$\frac{\varphi(b) - \varphi(a)}{b - a} = \varphi'(X), \qquad a < X < b \tag{2}$$

This theorem is known as the Law of the Mean.

Consider the curve $y = \varphi(x)$. The ratio $\frac{\varphi(b) - \varphi(a)}{b - a}$ is the slope of the line AB and the theorem asserts that there is at least one point C $(x = X)$ at which the tangent to the curve $y = \varphi(x)$ has the same slope. Thus the

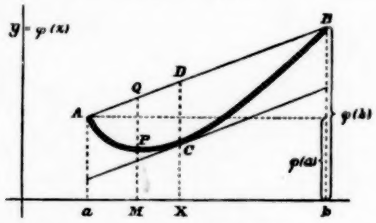


figure not only renders the theorem plausible, but it points the way to the analytic proof. For, the point C is also characterized by the condition that its distance CD from AB, measured along a line parallel to the y-axis, is a maximum. This distance for a variable point P is $PQ = MQ - MP$ or, since the equation of the right line through A, B is

$$y = \frac{\varphi(b) - \varphi(a)}{b - a}(x - a) + \varphi(a),$$

$$PQ = \Phi(x) = \frac{\varphi(b) - \varphi(a)}{b - a}(x - a) + \varphi(a) - \varphi(x).$$

The function $\Phi(x)$ vanishes when $x = a$ and when $x = b$, and it must therefore have a maximum or a minimum for at least one value X of x between a and b, i. e. its numerical value (the distance PQ) must have a maximum. Hence*

$$\Phi'(x) = \frac{\varphi(b) - \varphi(a)}{b - a} - \varphi'(X) = 0, \qquad a < X < b, \qquad \text{q. e. d.}$$

* For a detailed arithmetic proof of this last theorem cf. Tannery, § 135. The theorem is known as Rolle's theorem and stated with exactness is as follows: If $\Phi(x)$ is a single valued function of x, continuous in the interval $a \leqq x \leqq b$ and having a derivative at each point within the interval $a < x < b$; and if $\Phi(a) = 0$, $\Phi(b) = 0$; then there must be at least one value X of x within the interval for which

$$\Phi'(X) = 0, \qquad a < X < b.$$

Harnack uses the above figure in his *Differential Calculus* (§ 37) to illustrate the fact which the theorem asserts, but not to suggest a proof of the theorem. His statement of the theorem is incorrect.

7. *The Generalized Law of the Mean.* Let $f(x)$, $F(x)$ be any two functions satisfying the conditions imposed on $\varphi(x)$ and let $F(b) \neq F(a)$. Then

$$f(b) - f(a) = (b - a)f'(X_1), \qquad a < X_1 < b$$

$$F(b) - F(a) = (b - a)F'(X_2), \qquad a < X_2 < b$$

and

$$\frac{f(b) - f(a)}{F(b) - F(a)} = \frac{f'(X_1)}{F'(X_2)}.$$

Now this equation is in general not applicable to the treatment of the subject of §§ 11, 12, because there is no reason why X_1, X_2 should be equal to each other; it would be applicable if they were equal, and the question thus presents itself: Is it possible to find a value X such that, while perhaps different from both X_1 and X_2, still the equation

$$\frac{f(b) - f(a)}{F(b) - F(a)} = \frac{f'(X)}{F'(X)}, \qquad a < X < b$$

will hold? The answer to this question is contained in the Generalized Law of the Mean, which is:

THEOREM II. *Let $f(x)$, $F(x)$ be single valued continuous functions of x throughout the interval $a \leqq x \leqq b$, having derivatives for all values of x within this interval, $a < x < b$; and let the derivative of $F(x)$ be finite and different from 0, when x lies within this interval. Then there exists at least one value of X, mean between a and b, for which*

$$\frac{f(b) - f(a)}{F(b) - F(a)} = \frac{f'(X)}{F'(X)}, \qquad a < X < b. \tag{3}$$

Two lines of attack suggest themselves. One is to try to deduce this theorem from the foregoing by some ingenious transformation; the other, to study directly the content of the theorem and from the hypothesis to try to be led to a proof. Let us examine these two methods successively.

8. *First Proof.* Let $F(x) = z$, $x = F^{-1}(z)$,

$$f(x) = f[F^{-1}(z)] = \varphi(z),$$

and denote $F(a)$, $F(b)$ resp. by α, β. Then

$$\frac{f(b) - f(a)}{F(b) - F(a)} = \frac{\varphi(\beta) - \varphi(\alpha)}{\beta - \alpha} = \varphi'(Z), \qquad \alpha < Z < \beta.$$

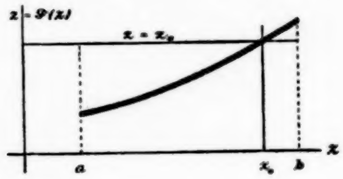
But

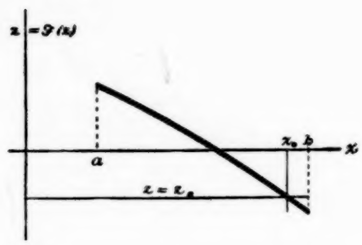
$$\varphi'(z) = \frac{d\varphi(z)}{dx} \cdot \frac{dx}{dz} = f'(x) \cdot \frac{1}{F'(x)}.$$

When x goes from a to b, z goes from α to β, and hence for some value X mean between a and b, $F(X) = Z$. Hence

$$\frac{f(b) - f(a)}{F(b) - F(a)} = \frac{f'(X)}{F'(X)}, \qquad a < X < b.$$

Thus the formal proof is complete. Were its steps justifiable? Consider





the function $z = F(x)$. Represent it by a curve. This function must always increase or else always decrease, when x goes from a to b.* For otherwise the curve would have points in which the tangent would be parallel to the x-axis (maxima, minima, etc.); but $F'(x)$ was not to vanish within the interval from a to b.—Hence we infer that the curve is cut by the line $z = z_0$, where $\alpha \leqq z_0 \leqq \beta$ in just one point; i. e. that x regarded as a function of z, $x = F^{-1}(z)$, is a single-valued function for all values of $\alpha \leqq z \leqq \beta$. That $F^{-1}(z)$ is continuous is obvious from the figure; and since

$$\lim_{\Delta z = 0} \frac{\Delta x}{\Delta z} = \frac{1}{\lim\limits_{\Delta x = 0} \frac{\Delta z}{\Delta x}}, \qquad \frac{dx}{dz} = \frac{1}{F'(x)},$$

a finite limit different from 0. Finally $f(x) = f[F^{-1}(z)]$ i. e. $\varphi(z)$ is a continuous function of z having a derivative given by the above formula for $\varphi'(z)$.

9. *Second Proof.* The second method was that employed by Cauchy.† The proof that follows is in its essential features the same as Cauchy's; but certain modifications in detail have permitted a gain in generality and simplicity.

* Such a function is called *monoton.*

† *Calcul différentiel*, quatrième leçon.

Let the derivatives $f'(x)$, $F'(x)$ be furthermore assumed to be continuous for all values of x within the interval: $a < x < b$. $F'(x)$ cannot change sign; suppose that

$$F'(x) > 0, \qquad a < x < b.$$

Now if the theorem were not true, we should have

$$\frac{f(b) - f(a)}{F(b) - F(a)} \gtrless \frac{f'(x)}{F'(x)}$$

for all values of $a < x < b$. Only one of the two signs however is possible; for if for some values of x the upper sign held, for other values, the lower, then the continuous function $f'(x)/F'(x)$ would pass through the intermediate value $\frac{f(b) - f(a)}{F(b) - F(a)}$ and for the corresponding value X of x the theorem would be true. Suppose then the upper sign were to hold:

$$\frac{f(b) - f(a)}{F(b) - F(a)} F'(x) - f'(x) > 0, \qquad a < x < b.$$

The left hand side of the inequality is the derivative of the function

$$\frac{f(b) - f(a)}{F(b) - F(a)} [F(x) - F(a)] - [f(x) - f(a)],$$

the constant of integration having been so determined as to make the function vanish when $x = a$. This derivative being positive, the function increases as x increases, and hence for $x = b$ must have a positive value. But for $x = b$ it vanishes. From this contradiction follows that the upper sign cannot hold. Similar reasoning applies to the lower sign, and hence the assumption that the theorem is not true when $F'(x) > 0$ is untenable. The case that $F'(x) < 0$ is treated in like manner, and hence the proof of the theorem is complete.

10. *Bonnet's Proof* is as follows. The theorem requires that the equation

$$\frac{f(b) - f(a)}{F(b) - F(a)} F'(x) - f'(x) = 0$$

should have a root X lying between a and b. The left hand side of this equation is the derivative of a function that is readily formed. Now try, in hopes of being able to apply Rolle's theorem, to construct this function in such a manner that it will vanish for $x = a$ and for $x = b$. The first of these conditions determines the function completely; it is

$$\frac{f(b) - f(a)}{F(b) - F(a)} [F(x) - F(a)] - [f(x) - f(a)]$$

and the function, as good luck will have it, vanishes for $x = b$ too. It satisfies all of the conditions of Rolle's theorem and hence its derivative vanishes for a mean value of x, or

$$\frac{f(b)-f(a)}{F(b)-F(a)}F'(X)-f'(X)=0\,,\qquad a<X<b\,. \quad \text{q. e. d.}$$

Presented in this form, which seems to have been the form in which Bonnet himself conceived it, Bonnet's proof is natural enough, though requiring somewhat more ingenuity than Cauchy's; and it is extremely simple. But as Bonnet's proof is usually presented* the function

$$\frac{f(b)-f(a)}{F(b)-F(a)}[F(x)-F(a)]-[f(x)-f(a)]$$

appears as a *deus ex machina*, for it is written down without the assignment of any reasons for its formation, which seems to have been the result of a happy guess. Such a mode of presentation is always unsatisfactory; but in the case of a theorem so fundamental as the foregoing, no pains ought to be spared in seeking out those lines of thought that led, or may naturally lead, to the source from which it sprang.

The Limits $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

11. *Proof of the Theorem of* § 2 *for Case* (1). It is assumed that the functions $f(x)$, $F(x)$ satisfy the conditions of Theorem II, § 7, for the interval from a to b, if b is not taken too far from a. $f(a) = 0$, $F(a) = 0$. Replace b by x. Then (3) becomes

$$\frac{f(x)}{F(x)}=\frac{f'(X)}{F'(X)},$$

where X lies between x and a and hence approaches a as its limit when x approaches a. If now $f'(x)/F'(x)$ approaches a limit when the *independent variable* x approaches a continuously (i. e. passing through all intermediate values), then $f'(X)/F'(X)$ must approach a limit, and this will be the same limit, when X approaches a, no matter whether X passes through *all* intermediate values or not.† Hence $f(x)/F(x)$ approaches a limit, namely $\lim\limits_{x=a}\frac{f'(x)}{F'(x)}$:

$$\lim_{x=a}\frac{f(x)}{F(x)} = \lim_{x=a}\frac{f'(x)}{F'(x)},$$

provided a is finite.

* Cf. Tannery, §§ 136, 137; Stolz, Ch. II.

† It is to be noticed that all we know about X is that it lies between x and a; hence when x approaches a, X, which is a function of x, may conceivably vary *discontinuously*, springing over whole intervals of values. The point of the above remark is that this makes no difference in the present case.

If $a = \infty$ and $f(\infty) = 0$, $F(\infty) = 0$, then the statement and proof of the theorem of § 7 can readily be modified to meet this case. It is a good exercise for the student to carry through this work. — Or a new variable $x' = \frac{1}{x}$ can be introduced and the proof given in the usual way.

12. *The Proof for Case* (2). Let $a = \infty$. Let $f(x)$, $F(x)$ be continuous functions of x for all values of $x \geqq B$, where B denotes a certain fixed value, and let *both** derivatives $f'(x)$, $F'(x)$ be different from 0 for all such values. Then it follows as in § 8 that $f(x)$, $F(x)$ are *monoton*. In (3) call b, x and a, x', and write the equation in the form:

$$\frac{f(x)}{F(x)} \cdot \frac{1 - \dfrac{f(x')}{f(x)}}{1 - \dfrac{F(x')}{F(x)}} = \frac{f'(X)}{F'(X)}, \qquad B \leqq x' < X < x .$$

B shall moreover be taken so large that $f(x)$, $F(x)$ remain numerically greater than unity when $x \geqq B$.

If x', x are allowed to increase without limit, X must also increase indefinitely and the right hand side of this equation will then by hypothesis approach a limit. Now $f(x)$, $f(x')$, $F(x)$, $F(x')$ all increase numerically at the same time indefinitely. If x can be made to increase so much faster than x' that $f(x')/f(x)$ and $F(x')/F(x)$ both approach 0, then the second factor on the left hand side will approach 1 and hence the first factor (I) must approach a limit and (II) this limit must coincide with $\lim\limits_{x=\infty} \frac{f'(x)}{F'(x)}$. The theorem will thus be established.

The proof turns then on showing that x', x can actually be made to increase indefinitely in such a manner that

$$\lim \frac{f(x')}{f(x)} = 0 , \quad \lim \frac{F(x')}{F(x)} = 0 .$$

Let x_1 be so taken as to satisfy the equation

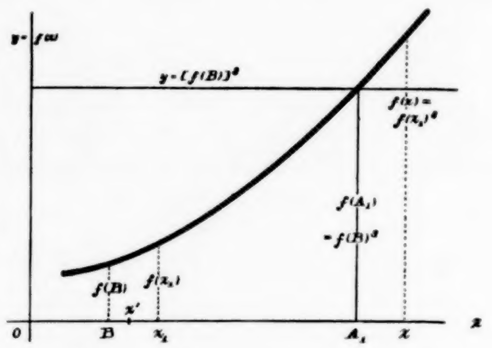
$$[f(x_1)]^3 = f(x) .$$

This will always be possible if x is taken sufficiently large. For $[f(B)]^3$ is numerically greater than $f(B)$ and hence the equation

$$[f(B)]^3 = f(x)$$

* The theorem is not stated here in the most general form to which the proof referred to in § 5 is applicable; but the present form is sufficiently general for practice.

will have a root A_1 greater than B; and if x is taken greater than A_1, $\sqrt[3]{f(x)}$



will be numerically greater than $f(B)$ and hence there will be an $x_1 > B$ such that

$$f(x_1) = \sqrt[3]{f(x)}\,.$$

Moreover if x' is any value of x lying between B and x_1, $f(x')$ will be numerically less than $f(x_1)$ and hence

$$\frac{[f(x')]^3}{f(x)} < 1\,.$$

From this inequality it follows that

$$\frac{f(x')}{[f(x)]^{\frac{1}{3}}} < 1 \quad \text{or} \quad \frac{f(x')}{f(x)} < \frac{1}{[f(x)]^{\frac{2}{3}}}\,.$$

The same reasoning applied to the function $F(x)$ shows that if A_2 is the root of the equation

$$[F(B)]^3 = F(x)$$

and x be taken larger than A_2, the equation

$$[F(x_2)]^3 = F(x)$$

will have a root x_2 greater than B; and if x' be taken between B and x_2,

$$\frac{[F(x')]^3}{F(x)} < 1\,, \quad \text{or} \quad \frac{F(x')}{F(x)} < \frac{1}{[F(x)]^{\frac{2}{3}}}\,.$$

Now let A denote the larger of the quantities A_1, A_2, x' the smaller of the quantities x_1, x_2. Then it follows from the foregoing that we shall have simultaneously

$$\frac{f(x')}{f(x)} \leqq \frac{1}{[f(x)]^{\frac{2}{3}}}\,, \qquad \frac{F(x')}{F(x)} \leqq \frac{1}{[F(x)]^{\frac{2}{3}}}\,, \qquad \text{if } x \geqq A\,.$$

Hence $f(x')/f(x)$ and $F(x')/F(x)$ both converge toward 0 when $x = \infty$, x' increasing at the same time indefinitely, and the proof of the theorem is complete.

13. *Applications. The Limits* $0 \times \infty$, 1^∞, 0^0, ∞^0. The general subject of the determination of a limit is of fundamental importance in analysis. The limits $0/0$, ∞/∞ together with those just enumerated are among the simplest and most common. In the differential calculus the computation of a small number of limits, like $\lim_{x=0} \frac{\sin x}{x}$ and $\lim_{x=\infty} \left[1 + \frac{1}{x}\right]^x$, suffices for the differentiation of the elementary functions. The formulas of differentiation once obtained and the Theorem of § 2 makes possible *without the further computation of any limits*, the direct determination of those limits of the form above cited which arise most frequently in practice.* Here then is a general method for the determination of such limits, simple in theory and simple in application. To establish the theory of this method rigorously and at the same time in a manner easily intelligible to students of elementary calculus was the object of the second part of this paper.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS., *January*, 1897.

* The form $\infty - \infty$ does not appear in the above list for the reason that it is usually determined most simply by the aid of infinite series.

CERTAIN INVARIANTS OF A QUADRANGLE BY PROJECTIVE TRANSFORMATION.

Dr. Edgar Odell Lovett, Princeton, New Jersey.

A geometrical configuration depending on m independent parameters has $m-r$ independent invariants by an r parameter Lie group of point transformations; these $m-r$ invariants are found by the integration of the complete system of simultaneous partial differential equations

$$X_1 f = 0 \ , \ \ldots , \ X_r f = 0 \, ,$$

where the $X_i f$ are the r independent infinitesimal transformations which generate the r parameter group.

The object of this note is to determine the invariants of a plane quadrangle, i. e. of a system of four points in the plane, by several projective groups of the plane.

1. The general projective group of the plane is an eight parameter group generated by the following eight independent infinitesimal transformations:

$$\boxed{p \quad q \quad xp \quad yq \quad xq \quad yp \quad x^2p + xyq \quad xyp + y^2q}$$

where

$$p \equiv \frac{\partial f}{\partial x} \, , \ q \equiv \frac{\partial f}{y} \, .$$

The quadrangle is a configuration depending on eight independent parameters, namely, the coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4); accordingly the quadrangle has no invariants by the general projective group. If, however, the quadrangle degenerates into a system of four collinear points, the anharmonic ratio of the four points is a known invariant by the general projective group. Hence, for this case, if we construct the determinant whose vanishing expresses the condition that the simultaneous system

$$\begin{aligned}
&\sum_1^4{}_i \frac{\partial \varphi}{\partial x_i} = 0 \, , \quad \sum_1^4{}_i \frac{\partial \varphi}{\partial y_i} = 0 \, , \quad \sum_1^4{}_i x_i \frac{\partial \varphi}{\partial x_i} = 0 \, , \\
&\sum_1^4{}_i y_i \frac{\partial \varphi}{\partial y_i} = 0 \, , \quad \sum_1^4{}_i x_i \frac{\partial \varphi}{\partial y_i} = 0 \, , \quad \sum_1^4{}_i y_i \frac{\partial \varphi}{\partial x_i} = 0 \, , \\
&\sum_1^4{}_i \left[x_i^2 \frac{\partial \varphi}{\partial x_i} + x_i y_i \frac{\partial \varphi}{\partial y_i} \right] = 0 \, , \quad \sum_1^4{}_i \left[x_i y_i \frac{\partial \varphi}{\partial x_i} + y_i^2 \frac{\partial \varphi}{\partial y_i} \right] = 0 \, ,
\end{aligned} \tag{1}$$

have a solution, we find the following theorem:

If the four points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) *are collinear the determinant*

$$D = \begin{vmatrix} x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 & y_1^2 & y_2^2 & y_3^2 & y_4^2 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 \\ 0 & 0 & 0 & 0 & x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_1 & y_2 & y_3 & y_4 \\ x_1 & x_2 & x_3 & x_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{vmatrix} \tag{2}$$

vanishes.

2. There is no seven parameter projective group in the plane. There are two six parameter projective groups. The first of these is the general linear group

$$\begin{vmatrix} p & q & xp & yp & xq & yq \end{vmatrix}. \tag{a}$$

The invariants sought are the solutions of the complete system of partial differential equations

$$\sum_1^4{}_i \frac{\partial\varphi}{\partial x_i} = \sum_1^4{}_i \frac{\partial\varphi}{\partial y_i} = \sum_1^4{}_i x_i \frac{\partial\varphi}{\partial x_i} = \sum_1^4{}_i y_i \frac{\partial\varphi}{\partial x_i} = \sum_1^4{}_i x_i \frac{\partial\varphi}{\partial y_i} = \sum_1^4{}_i y_i \frac{\partial\varphi}{\partial y_i} = 0\,. \tag{3}$$

The first and second of these equations assert that the invariant functions $\varphi(x_i, y_i)$ depend on the differences

$$u_1 = x_2 - x_1\,, \; u_2 = x_3 - x_1\,, \; u_3 = x_4 - x_1\,;$$

$$v_1 = y_2 - y_1\,, \; v_2 = y_3 - y_1\,, \; v_3 = y_4 - y_1\,;$$

so that the third and sixth may be written, respectively,

$$\sum_1^3{}_i u_i \frac{\partial\varphi}{\partial u_i} = 0\,, \; \sum_1^3{}_i v_i \frac{\partial\varphi}{\partial v_i} = 0\,; \tag{4}$$

and the fourth and fifth become, respectively,

$$\sum_1^3{}_i v_i \frac{\partial\varphi}{\partial u_i} = 0\,, \; \sum_1^3{}_i u_i \frac{\partial\varphi}{\partial v_i} = 0\,. \tag{5}$$

The equations (4) are equivalent to the simultaneous systems

$$\frac{du_1}{u_1} = \frac{du_2}{u_2} = \frac{du_3}{u_3}; \quad \frac{dv_1}{v_1} = \frac{dv_2}{v_2} = \frac{dv_3}{v_3},$$

which show that φ is a function only of the variables

$$s_1 = \frac{u_2}{u_1}, \quad s_2 = \frac{u_3}{u_1}, \quad t_1 = \frac{v_2}{v_1}, \quad t_2 = \frac{v_3}{v_1}.$$

If these be introduced as new variables in equations (5), the resulting equations show that the invariant function is a function of $(s_1 - t_1)/(s_2 - t_2)$.

Developing $(s_1 - t_1)/(s_2 - t_2)$ and its analogous forms we find the three invariant equations

$$| u_1, v_2 | / | u_1, v_3 | = c_1, \quad | u_1, v_2 | / | u_2, v_3 | = c_2, \quad | u_1, v_3 | / | u_2, v_3 | = c_3,$$

only two of which are independent,* as was to be expected since an invariant function of eight independent variables by a six parameter group was sought. In order to interpret these invariants geometrically we develop c_1, c_2, and c_3 still further. In fact

$$c_1 = | u_1, v_2 | / | u_1, v_3 | = \frac{(x_3 - x_1)(y_2 - y_1) - (x_2 - x_1)(y_3 - y_1)}{(x_4 - x_1)(y_2 - y_1) - (x_2 - x_1)(y_4 - y_1)} = \frac{\triangle 123}{\triangle 124};$$

similarly

$$c_2 = \frac{\triangle 132}{\triangle 134}, \quad c_3 = \frac{\triangle 142}{\triangle 143}.$$

Hence the theorem :

If the quadrangle (1234) *be transformed by the* LIE *group*

$$\boxed{p \quad q \quad xp \quad yp \quad xq \quad yq}$$

the ratios of the triangle (123) *to the triangles* (124) *and* (134) *remain unchanged; i. e. the area of the quadrangle is changed in a constant ratio.*

---

* That there are not more than two independent solutions of the original complete system (3) is seen from the fact that all sixth order determinants of the matrix

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_1 & y_2 & y_3 & y_4 \\ y_1 & y_2 & y_3 & y_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_1 & x_2 & x_3 & x_4 \end{vmatrix}$$

do not vanish identically.

It is of course obvious that any function of c_1, c_2, and c_3 is invariant by the transformations of the group. The result of this article might have been anticipated. The group is the so-called general linear group of the plane whose characteristic property is that by it all areas are changed in the same ratio.

3. The only other six parameter projective group in the plane is the group

$$\boxed{xp \quad yp \quad xq \quad yq \quad x^2p + xyq \quad xyp + y^2q} \,. \tag{β}$$

The corresponding complete system to be solved is

$$\begin{gathered} \sum_1^4{}_i x_i \frac{\partial\varphi}{\partial x_i} = \sum_1^4{}_i y_i \frac{\partial\varphi}{\partial x_i} = \sum_1^4{}_i x_i \frac{\partial\varphi}{\partial y_i} = \sum_1^4{}_i y_i \frac{\partial\varphi}{\partial y_i} = 0\,, \\ \sum_1^4{}_i \left[x_i^2 \frac{\partial\varphi}{\partial x_i} + x_i y_i \frac{\partial\varphi}{\partial y_i} \right] = \sum_1^4{}_i \left[x_i y_i \frac{\partial\varphi}{\partial x_i} + y_i^2 \frac{\partial\varphi}{\partial y_i} \right] = 0\,. \end{gathered} \tag{6}$$

The first and fourth of these equations give φ as a function of the variables

$$\bar{u}_1 = \frac{x_2}{x_1}\,, \quad \bar{u}_2 = \frac{x_3}{x_1}\,, \quad \bar{u}_3 = \frac{x_4}{x_1}\,; \quad \bar{v}_1 = \frac{y_2}{y_1}\,, \quad \bar{v}_2 = \frac{y_3}{y_1}\,, \quad \bar{v}_3 = \frac{y_4}{y_1}\,.$$

These introduced as new variables reduce the second and third equations respectively to

$$\sum_1^3{}_i (\bar{u}_i - \bar{v}_i) \frac{\partial\varphi}{\partial u_i} = 0\,, \quad \sum_1^3{}_i (\bar{u}_i - \bar{v}_i) \frac{\partial\varphi}{\partial v_i} = 0\,; \tag{7}$$

and the fifth and sixth respectively to

$$\begin{gathered} \sum_1^3{}_i \left\{ \bar{u}_i (\bar{u}_i - 1) \frac{\partial\varphi}{\partial u_i} + \bar{v}_i (\bar{u}_i - 1) \frac{\partial\varphi}{\partial v_i} \right\} = 0\,, \\ \sum_1^3{}_i \left\{ \bar{u}_i (\bar{v}_i - 1) \frac{\partial\varphi}{\partial u_i} + \bar{v}_i (\bar{v}_i - 1) \frac{\partial\varphi}{\partial v_i} \right\} = 0\,. \end{gathered} \tag{8}$$

The equations (8) are satisfied if φ is a function of the variables

$$r_1 = \frac{\bar{u}_1}{\bar{v}_1}\,, \quad r_2 = \frac{\bar{u}_2}{\bar{v}_2}\,, \quad r_3 = \frac{\bar{u}_3}{\bar{v}_3}\,;$$

these introduced in equations (7) reduce them to the forms respectively

$$\sum_1^3{}_i (r_i - 1) \frac{\partial\varphi}{\partial r_i} = 0\,, \quad \sum_1^3{}_i r_i (r_i - 1) \frac{\partial\varphi}{\partial r_i} = 0\,, \tag{9}$$

the first of which admits of the solutions

$$\frac{r_2 - 1}{r_1 - 1} = s_1, \quad \frac{r_3 - 1}{r_1 - 1} = s_2,$$

which substituted in the second produce the general solution

$$\frac{s_2 - 1}{s_1 - 1} = \text{constant}.$$

Expanding this last expression there results

$$\frac{s_2 - 1}{s_1 - 1} = \frac{r_3 - r_1}{r_2 - r_1} = \frac{\bar{u}_2}{\bar{v}_2} \frac{|\bar{u}_3, \bar{v}_1|}{|\bar{u}_2, \bar{v}_1|} = \frac{y_3}{y_4} \frac{|x_4, y_2|}{|x_3, y_2|} = \frac{y_3 \triangle 024}{y_4 \triangle 023},$$

where zero is the origin. Hence the theorem

If a quadrangle (1234) *is subjected to the transformations of the* Lie *group*

$$\left| xp \quad yp \quad xq \quad yq \quad x^2p + xyq \quad xyp + y^2q \right|$$

the ratio of the triangles (023) *and* (024) *varies as the ratio of the ordinates* y_3 *and* y_4*; similarly the ratio of the triangles* (012) *and* (013) *varies as the ratio of the abscissas* x_2 *and* x_3*; these two factors of proportionality are the independent invariants.*

That there are no more than two independent invariants may be verified by constructing the test matrix as in the foot-note to the preceding section and observing that all sixth order determinants do not vanish.

4. There are three types of five parameter projective groups of the plane. We shall take space to find the invariants of the quadrangle by two of them because of the beauty of the geometrical results. The group

$$\left| p \quad q \quad xq \quad xp - yq \quad yp \right| \qquad (\lambda)$$

yields the complete system

$$\sum_1^4{}_i \frac{\partial\varphi}{\partial x_i} = \sum_1^4{}_i \frac{\partial\varphi}{\partial y_i} = \sum_1^4{}_i x_i \frac{\partial\varphi}{\partial y_i} = \sum_1^4{}_i \left[x_i \frac{\partial\varphi}{\partial x_i} - y_i \frac{\partial\varphi}{\partial y_i} \right] = \sum_1^4{}_i y_i \frac{\partial\varphi}{\partial x_i} = 0. \quad (10)$$

This system of five equations in eight variables has three independent solutions but no more as its matrix shows.

The first two equations make φ a function of the differences

$$u_1 = x_2 - x_1, \quad u_2 = x_3 - x_1, \quad u_3 = x_4 - x_1;$$

$$v_1 = y_2 - y_1, \quad v_2 = y_3 - y_1, \quad v_3 = y_4 - y_1;$$

and these substituted in the last three equations lead to the three independent solutions

$$|u_1, v_2| = 2\triangle 123 = c_1, \quad |u_1, v_3| = 2\triangle 124 = c_2, \quad |u_2, v_3| = 2\triangle 134 = c_3;$$

whence the theorem

The area of the quadrangle (1234) *is left invariant by the* LIE *group*

$$\boxed{p \quad q \quad xq \quad xp - yq \quad yp} \, .$$

This invariant might have been anticipated had it been observed that the group is the so-called special linear group whose peculiar property is that of leaving all areas invariant.

5. Consider finally the five parameter group

$$\boxed{xq \quad xp - yq \quad yp \quad x^2p + xyq \quad xyp + y^2q} \qquad (\mu)$$

the simultaneous system to be satisfied by the invariant functions sought is made up of the following equations:

$$\sum_1^4{}_i x_i \frac{\partial\varphi}{\partial y_i} = \sum_1^4{}_i \left[x_i \frac{\partial\varphi}{\partial x_i} - y_i \frac{\partial\varphi}{\partial y_i} \right] = \sum_1^4{}_i y_i \frac{\partial\varphi}{\partial x_i} = 0\,,$$

$$\sum_1^4{}_i \left[x_i^2 \frac{\partial\varphi}{\partial x_i} + x_i y_i \frac{\partial\varphi}{\partial y_i} \right] = \sum_1^4{}_i \left[x_i y_i \frac{\partial\varphi}{\partial x} + y_i^2 \frac{\partial\varphi}{\partial y_i} \right] = 0\,.$$

The first and third demand the φ be a function of the variables

$$\bar{u}_1 = \frac{x_2}{x_1}, \ \bar{u}_2 = \frac{x_3}{x_1}, \ \bar{u}_3 = \frac{x_4}{x_1}; \quad \bar{v}_1 = \frac{y_2}{y_1}, \ \bar{v}_2 = \frac{y_3}{y_1}, \ \bar{v}_3 = \frac{y_4}{y_1}.$$

These are found to satisfy the second equation identically and to reduce the fourth and fifth respectively to

$$\sum_1^3{}_i \left\{ \bar{u}_i(\bar{u}_i - 1) \frac{\partial\varphi}{\partial u_i} + \bar{v}_i(\bar{u}_i - 1) \frac{\partial\varphi}{\partial v_i} \right\} = 0\,,$$

$$\sum_1^4{}_i \left\{ \bar{u}_i(\bar{v}_i - 1) \frac{\partial\varphi}{\partial u_i} + \bar{v}_i(\bar{v}_i - 1) \frac{\partial\varphi}{\partial v_i} \right\} = 0\,.$$

Whence the solutions

$$\frac{\bar{u}_1}{\bar{v}_1}, \ \frac{\bar{u}_2}{\bar{v}_2}, \ \frac{\bar{u}_3}{\bar{v}_3}.$$

Hence the theorem

If the quadrangle **(1234)** *is operated on by the transformations of the* Lie *group*

$$\boxed{xq \quad xp - yq \quad yp \quad x^2p + xyq \quad xyp + y^2q}$$

the ratio of the abscissas of any two points varies as the ratio of the ordinates of the two points; the factor of proportionality is different for different pairs.

The reader may find it interesting to construct the other invariants of the quadrangle by the remaining projective subgroups of the plane. He will observe that many of the results admit of immediate extension to a system of n points. Lie has determined all projective groups of the plane; they are tabulated in Chapter XI, § 4, of his Lectures on Continuous Groups.

6. The corresponding theorems for the quadrilateral may be found in the following manner.

An infinitesimal transformation in point coordinates is designated symbolically by

$$Uf \equiv \xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y};$$

by this transformation x and y receive the increments

$$\delta x = \xi(x, y)\,\delta t, \quad \delta y = \eta(x, y)\,\delta t.$$

Let its equivalent transformation in line coordinates be

$$Vf \equiv \varphi(u, v)\frac{\partial f}{\partial u} + \psi(u, v)\frac{\partial f}{\partial v};$$

by which u and v are respectively increased by

$$\delta u = \varphi(u, v)\,\delta t, \quad \delta v = \psi(u, v)\,\delta t.$$

Either of these transformations may be derived from the other by the conditions

$$ux + vy + 1 = 0,$$

$$\delta(ux + vy + 1) = 0.$$

Putting $p = \frac{\partial f}{\partial x}$, $q = \frac{\partial f}{\partial y}$, $s = \frac{\partial f}{\partial u}$, $t = \frac{\partial f}{\partial v}$, the following table explains its own construction:

$\xi p + \eta q$	$\delta(ux + vy + 1) = 0$	$\varphi s + \psi t$
p	$u + x\delta u + y\delta v = 0$	$u^2 s + uvt$
q	$v + x\delta u + y\delta v = 0$	$uvs + v^2 t$
xp	$ux + x\delta u + y\delta v = 0$	$-us$
xq	$vx + x\delta u + y\delta v = 0$	$-vs$
yp	$uy + x\delta u + y\delta v = 0$	$-ut$
yq	$vy + x\delta u + y\delta v = 0$	$-vt$
$x^2 p + xyq$	$ux^2 + vxy + x\delta u + y\delta v = 0$	s
$xyp + y^2 q$	$uxy + vy^2 + x\delta u + y\delta v = 0$	t

The determinant D, written in (u_1, v_1), (u_2, v_2), (u_3, v_3), (u_4, v_4) in place of the x, y, vanishes if the four lines are concurrent, since by the above, the general projective group is seen to have the same form in line coordinates that it has in point coordinates.

The other theorems can be immediately translated by observing that the groups (α), (β), (λ), (μ) in point coordinates assume respectively the forms (β), (α), (μ), (λ) in line coordinates, and the corresponding geometrical interpretations offer no difficulty.

It is perhaps not out of place to call attention to the fact that Lie expresses the property that a group is its own dualistic by enclosing the transformations of the group in a double rectangle thus

$$\boxed{\boxed{p \quad q \quad xp \quad xq \quad yp \quad yq \quad x^2p + xyq \quad xyp + y^2q}}\,.$$

PRINCETON, NEW JERSEY, 10 *January*, 1898.

THE GENERAL TRANSFORMATION OF THE GROUP OF EUCLIDIAN MOVEMENTS.

By Prof. J. M. Page, Charlottesville, Va.

One of the most important of the transformation groups that occur in the Theory of Differential Equations, or in Geometry, is the so-called group of Euclidian Movements. We propose to investigate the *form* of the most general transformation of this group in the plane and in space.

— § 1. —

The infinitesimal transformations of the group of movements in the plane are $\left[\text{writing } p \text{ for } \frac{\partial f}{\partial x}, \text{ and } q \text{ for } \frac{\partial f}{\partial y}\right]$,

$$p, \quad q, \quad yp - xq,$$

the first two being the translations, and the third the rotation. The most general infinitesimal transformation of this G_3 (or three-fold group), has the form

$$Uf \equiv Ap + Bq + C(yp - xq); \quad A, B, C, \text{ consts.},$$

and since Uf and *const.* Uf may be considered as *equivalent* infinitesimal transformations, there are clearly ∞^2 infinitesimal transformations in all in this G_3.

We know that points which are absolutely invariant under the transformation

$$Uf \equiv (A + Cy)p + (B - Cx)q$$

are obtained by writing

$$A + Cy = 0, \quad B - Cx = 0, \qquad (C \neq 0).$$

Hence, the only invariant point (within a finite distance of the origin) is

$$x = \frac{B}{C}, \quad y = -\frac{A}{C}.$$

In order to find the simplest form for the general infinitesimal transformation of the G_3 when the variables are referred to rectangular axes, let us transform the origin to the point $\frac{B}{C}, -\frac{A}{C}$ by introducing the new variables

$$x' = x - \frac{B}{C}, \quad y' = y + \frac{A}{C}.$$

We may assume $C \neq 0$, otherwise Uf would be a mere translation : hence, in the new variables Uf becomes

$$U'f = U(x')\frac{\partial f}{\partial x'} + U(y')\frac{\partial f}{\partial y'}$$

$$= U(x)\frac{\partial f}{\partial x'} + U(y)\frac{\partial f}{\partial y'}$$

$$= (A + Cy)\frac{\partial f}{\partial x'} + (B - Cx)\frac{\partial f}{\partial y'}$$

$$= C\left[y'\frac{\partial f}{\partial x'} - x'\frac{\partial f}{\partial y'}\right];$$

or as we may write it,

$$U'f = y'\frac{\partial f}{\partial x'} - x'\frac{\partial f}{\partial y'}.$$

Hence we see that by a proper choice of the rectangular coordinate axes, *the most general transformation of the G_3 of movements in the plane may be written in the form of a mere rotation.*

The path-curves of $U'f$ are given by

$$\frac{dx'}{y'} = -\frac{dy'}{x'},$$

and hence they have the form

$$x'^2 + y'^2 = c^2; \qquad (c = \text{const.})$$

so that the path-curves of Uf are given by

$$\left[x - \frac{B}{C}\right]^2 + \left[y + \frac{A}{C}\right]^2 = c^2.$$

Since the finite equations of any transformation

$$Uf = \xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y}$$

are obtained by integrating the simultaneous system

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dt,$$

it is clear that when the equation

$$\frac{dx}{\xi} = \frac{dy}{\eta}$$

has been integrated, the finite equations of the infinitesimal transformation

may be obtained by a *quadrature*. Hence, the finite equations of the transformation

$$Uf = (A + Cy)\frac{\partial f}{\partial x} + (B - Cx)\frac{\partial f}{\partial y}$$

may now be obtained by a quadrature.

If we indicate the *extended* transformation corresponding to Uf by

$$U^{(n)}f = \xi(x, y)p + \eta(x, y)q + \eta'(x, y, y')q' + \ldots + \eta^{(n)}(x, y, y', y'' \ldots y^{(n)})q^{(n)}$$

where

$$y^{(i)} = \frac{d^i y}{dx^i} \text{ and } q^{(i)} = \frac{\partial^i f}{\partial y},$$

the *Differential Invariants* of the transformation Uf may be found by integrating the simultaneous system

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dy'}{\eta'} = \ldots = \frac{dy^{(n)}}{\eta^{(n)}}.$$

Prof. Lie has shown* that when we know the path-curves of the transformation, i. e. when the equation

$$\frac{dx}{\xi} = \frac{dy}{\eta}$$

has been integrated, the other integral-functions of the simultaneous system may be found by one quadrature and a finite number of differentiations.

Thus we may consider the Differential Invariants of the transformation

$$Uf = (A + By)p + (B - Cx)q$$

to be all known: and from this it follows that the whole Theory of Invariance of the G_3 of movements in the plane, including the theory of the congruence of plane curves, may be derived by operations which do not involve any further integration. This theory has been worked out in detail by Prof. Lie† in a different manner.

These results are very simple for the case considered above, but they are given here merely to indicate the manner in which such questions are to be treated.

* Compare "Page's Differential Equations," p. 69.

† "Continuirliche Gruppen," Chap. 22.

— § 2. —

The group of movements in space consists of the six infinitesimal transformations

$$p, \quad q, \quad r, \quad yp - xq, \quad zq - yr, \quad xr - zp,$$

the first three being the translations, and the other three the rotations.

The most general infinitesimal transformation of the G_6 has the form

$$Uf = Ap + Bq + Cr + D(yp - xq) + E(zq - yr) + F(xr - zp),$$
$$A, B, \ldots F \text{ consts.}$$

Since Uf and ρUf, where $\rho =$ const., are equivalent infinitesimal transformations, we may choose the above transformation in the form ρUf, where

$$\rho = \frac{1}{\sqrt{D^2 + E^2 + F^2}}.$$

It is clear that this is equivalent to considering the constants D, E, F to be connected by the relation

$$D^2 + E^2 + F^2 = 1: \tag{1}$$

and the reason for this assumption will appear from the sequel.

We may write Uf in the form

$$Uf = (A + Dy - Fz)p + (B - Dx + Ez)q + (C - Ey + Fx)r:$$

and any invariant points, or curves consisting of absolutely invariant points, are obtained by writing

$$\left.\begin{aligned} A + Dy - Fz &= 0, \\ B - Dx + Ez &= 0, \\ C - Ey + Fx &= 0, \end{aligned}\right\}. \tag{2}$$

Multiplying these equations in order by E, F, D, and adding, we have

$$AE + BF + CD = 0. \tag{3}$$

Since $A, B, \ldots F$ are undetermined constants, except for the relation (1), (3) will usually not be a true equation. We shall, however, divide our problem into two parts, by assuming, first, that (3) holds; and, secondly, that (3) does not hold.

I: We assume that the constants $A, B, \ldots F$ have been chosen in such manner that

$$AE + BF + CD = 0. \tag{3}$$

It is then clear that the equations (2) are not independent; and hence any two of them, say

$$\left.\begin{aligned} Fx - Ey + C &= 0 \\ Dx - Ez - B &= 0 \end{aligned}\right\} \tag{4}$$

determine a straight line which is invariant under Uf in such manner that each point on the line is separately invariant.

The line (4) cuts the xy-plane at the point

$$B/D, \; -A/D \qquad (D \neq 0)$$

and we choose this point as the origin by writing

$$x' = x - B/D, \quad y' = y + A/D, \quad z' = z.$$

The line (4) then becomes

$$\left.\begin{aligned} Fx' - Ey' &= 0 \\ Dx' - Ez' &= 0 \end{aligned}\right\} \tag{5}$$

while Uf assumes the form

$$\begin{aligned} U'f &= U(x')\frac{\partial f}{\partial x'} + U(y')\frac{\partial f}{\partial y'} + U(z')\frac{\partial f}{\partial z'} \\ &= U(x)\frac{\partial f}{\partial x'} + U(y)\frac{\partial f}{\partial y'} + U(z)\frac{\partial f}{\partial z'} \\ &= (A + Dy - Fz)\frac{\partial f}{\partial x'} + (B - Dx + Ez)\frac{\partial f}{\partial y'} + (C - Ey + Fx)\frac{\partial f}{\partial z'} \\ &= (Dy' - Fz')\frac{\partial f}{\partial x'} + (Ez' - Dx')\frac{\partial f}{\partial y'} + (Fx' - Ey')\frac{\partial f}{\partial z'}. \end{aligned}$$

This transformation is now made up of the three rotations; and it was geometrically clear *a priori* that the three translations must disappear when the line (4) is invariant, since they leave no point within a finite distance of the origin invariant.

We shall now choose the invariant line (5) as the z-axis. This line is also represented by the equations

$$Fx' - Ey' = 0,$$

$$Fx' - Ey' - \frac{E^2 + F^2}{FD}(Dx' - Ez') = 0,$$

which may be written:

$$\left.\begin{aligned} &Fx' - Ey' = 0, \\ &EDx' + FDy' - (E^2 + F^2)z' = 0. \end{aligned}\right\} \tag{6}$$

The planes represented by (6) are now perpendicular; and a third plane through the origin perpendicular to these two is given by

$$Ex' + Fy' + Dz' = 0\,.$$

Let us now introduce new variables by writing

$$X = Fx' - Ey'\,,\quad Y = EDx' + FDy' - (E^2 + F^2)\,z'\,,$$
$$Z = Ex + Fy' + Dz'\,.$$

We then have:

$$U'(X) = Y,\quad U'(Y) = -(E^2 + F^2 + D^2)X,\quad U'(Z) = 0\,;$$

or, on account of (1),

$$U'f = U(X)\frac{\partial f}{\partial x} + U(Y)\frac{\partial f}{\partial y} + U(Z)\frac{\partial f}{\partial z}$$
$$= Y\frac{\partial f}{\partial x} - X\frac{\partial f}{\partial y}.$$

We see that this is an infinitesimal rotation in the variables X, Y, Z. Hence, *when the relation* (3) *exists, the most general movement possible in space is a rotation.* Geometrically, this was *a priori* evident; for when a straight line, consisting of invariant points, is invariant, it is clear that the only movement possible for the other points in space is a rotation around the invariant line.

II. Let us now suppose that no relation of the form

$$AE + BF + CD = 0$$

exists.

As in the last case, let us transform the origin to the point B/D, $-A/D$, by writing

$$x' = x - \frac{B}{D},\quad y' = y + \frac{A}{D},\quad z' = z\,. \qquad (D \neq 0)$$

The transformation Uf becomes

$$U'f = (Dy' - Fz')\frac{\partial f}{\partial x'} + (Ez' - Dx')\frac{\partial f}{\partial y'} + (Fx' - Ey' + \varkappa)\frac{\partial f}{\partial z'},$$

where

$$\varkappa = \frac{AE + BF + DC}{D}.$$

Now introduce the new independent variables X, Y, Z by means of:

$$X = Fx' - Ey' + \varkappa\,,\quad Y = EDx' + FDy' - (E^2 + F^2)z'\,,$$
$$Z = Ex' + Fy' + Dz'\,;$$

and we find

$$U'f = U(X)\frac{\partial f}{\partial x} + U(Y)\frac{\partial f}{\partial y} + U(Z)\frac{\partial f}{\partial z}$$

$$= Y\frac{\partial f}{\partial x} - (X - \varkappa D^2)\frac{\partial f}{\partial y} + D\varkappa\frac{\partial f}{\partial z}.$$

Finally, introduce as new variables x_1, y_1, z_1, by means of

$$x_1 = X - \varkappa D^2, \quad y_1 = Y, \quad z_1 = Z/D\varkappa. \qquad (D \neq 0)$$

Hence the transformation assumes the form

$$U_1 f = y_1\frac{\partial f}{\partial x_1} - x_1\frac{\partial f}{\partial y_1} + \frac{\partial f}{\partial z_1}.$$

Hence, *the most general transformation of the group of movements in space*

$$Uf = (A + Dy - Fz)p + (B - Dx + Ez)q + (C - Ey + Fx)r,$$

where the constants satisfy the relation

$$E^2 + F^2 + D^2 = 1,$$

may, by a proper choice of rectangular axes, be written in the canonical form

$$Uf = yp - xq + r.$$

This transformation is evidently made up of a rotation around the z-axis, combined with a translation along that axis, i. e. all the points in space move on helixes wound upon right cylinders of which the z-axis is the axis. Hence, we have a result well known in kinematics that *the most general movement of a point in space is equivalent to a movement of that point on a helix.*

The aggregate of the transformations performed above upon Uf is clearly represented by

$$\left.\begin{aligned} x_1 &= F\left[x - \frac{B}{D}\right] - E\left[y + \frac{A}{D}\right] + \varkappa(1 - D^2), \\ y_1 &= ED\left[x - \frac{B}{D}\right] + FD\left[y + \frac{A}{D}\right] - (E^2 + F^2)z, \\ z_1 &= \left\{E\left[x - \frac{B}{D}\right] + F\left[y + \frac{A}{D}\right] + Dz\right\} \div D\varkappa, \end{aligned}\right\} \qquad (7)$$

and these equations enable us to pass from the first form of the infinitesimal transformation to the last at one step.

In the form

$$U_1 f = y_1 p_1 - x_1 q_1 + r_1$$

it is obvious that the path-curves of the transformation are helixes; and they are given by the integrals of the simultaneous system:

$$\frac{dx_1}{y_1} = -\frac{dy_1}{x_1} = \frac{dz_1}{1},$$

in the form

$$x_1^2 + y_1^2 = c_1^2, \quad \tan^{-1}\frac{x_1}{y_1} - z_1 = c_2. \tag{8}$$

In order to obtain the path-curves of the transformation in its original form, we only need substitute in (8) the values of x_1, y_1, z_1, given by (7).

Since $U_1 f$ leaves the straight line $x_1 = y_1 = 0$ invariant, while the points on this line are interchanged among each other, it is clear that the original transformation always leaves the straight line

$$\left.\begin{aligned} &F(x - B/D) - E\left[y + \frac{A}{D}\right] + z(1 - D^2) = 0 \\ &ED(x - B/D) + FD(y + A/D) - (E^2 + F^2)z = 0 \end{aligned}\right\}$$

similarly invariant:—and it is geometrically evident *a priori* that since the most general infinitesimal movement must always leave a straight line invariant, the most general movement to which a point of general position of space can be subjected is a movement on a helix.

It is clear that we may now obtain the finite equations of the original transformation by a quadrature. We know the *Invariants* (8) of the transformation; and if we *extend* the transformation under the hypothesis that x, y, and z are connected by a relation of the form

$$z = f(x, y)$$

we may, according to a theorem of Lie, find all the Differential Invariants of the G_6 of movements of the form

$$\Omega(x, y, z, p, q, r, s, t \ldots)$$

and establish the theory of the congruence of *surfaces* in space, *without further integration:* and a similar remark is true of *curves*, i. e. when x, y, z are connected by the relations

$$z = \varphi(x), \quad y = \psi(x).$$

These theories have been developed by Lie, in detail, from another standpoint.

CONCOMITANT BINARY FORMS IN TERMS OF THE ROOTS.*

By Miss Annie L. MacKinnon, Aurora, N. Y.

Tables of the irreducible Covariants and Invariants of the lower quantics including the Sextic, and of pairs of the first four quantics (including the linear quantic).

The arrangement of the following Tables is that described in Vol. IX, p. 116. It should be noticed that the symbol Σ used in Tables V–X is a symbol of double summation, so that the root expressions are symmetric with regard to the primed roots and also symmetric with regard to the unprimed roots; thus, $\Sigma(\alpha\alpha')\beta\beta' = (\alpha\alpha')\beta\beta' + (\alpha\beta')\beta\alpha' + (\beta\alpha')\alpha\beta' + (\beta\beta')\alpha\alpha'$. In the Tables attention is called to the *unique* forms, where a *unique* form is understood to be a form which in its system is the only covariant of its order, weight and degree. The unique forms are of special interest in that they may be derived from any transvectant of the given order, weight and degree. It should be emphasized that the particular root symbol given for any particular covariant or invariant is often not the only root symbol that might be given for that form and that this is true not only among the unique forms. A simple example will make clear that different root symbols may exist for the same form; thus the terms $(\alpha\beta)(\alpha\gamma)\beta\gamma\delta^2\epsilon^2$ and $(\alpha\beta)^2\gamma^2\delta^2\epsilon^2$ occur in the development of $\{ff'\}^2$ of the Quintic and $\Sigma(\alpha\beta)(\alpha\gamma)\beta\gamma\delta^2\epsilon^2$ and $\Sigma(\alpha\beta)^2\gamma^2\delta^2\epsilon^2$ are both root symbols for H of the Quintic, since this is an irreducible form in the system of the Quintic and $\{ff'\}^2$ is its transvectant.

In the transvectants in the systems of the single quantics, the quantic itself is denoted by f. In the systems of the pairs of quantics, the first mentioned quantic is denoted by $f_{(1)}$ and all of its invariants and covariants have the subscript (1), and the second quantic is denoted by $f_{(2)}$ and all of its invariants and covariants have the subscript (2). Whenever there occurs a transvectant of any covariant over itself the letter denoting the second covariant is primed; thus in $\{ff'\}^2$, $f = f'$, and in $\{HH'\}^2$, $H = H'$.

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* This paper is supplementary to that which appeared in Nos. 4 and 5 of Vol. IX, and contains the Tables to which reference was made.

TABLE I.

No.	Name.	Cayley Symbol.	A. C. Symbol.	Root Symbol.	Root Differences.	Transvectant.	Remarks.
				QUADRIC.			
1	D	$\overline{12}^2$	$(ab)^2$	$(\alpha\beta)^2$	$(\alpha-\beta)^2$	$\{ff'\}^2$	See H in cubic, quartic, etc.
				CUBIC.			
1*	$H(\Delta)$	$\overline{12}^2$	$(ab)^2ab$	$\Sigma(\alpha\beta)^2\gamma^2$	$\Sigma(\alpha-\beta)^2(x-\gamma)^2$	$\{ff'\}^2$	See pp. 106, 138
2	$J(Q)$	$\overline{12}^2 . \overline{13}$	$(ab)^2(ac)bc^2$	$\Sigma(\alpha\beta)^2(\alpha\gamma)\beta\gamma^2$	$\Sigma(\alpha-\beta)^2(\alpha-\gamma)(x-\beta)(x-\gamma)^2$	$\{fH\}$	See Art. 45
3	$D(R)$	$\overline{12}^2 . \overline{34}^2 . \overline{13} . \overline{24}$	$(ab)^2(cd)(ac)(bd)$	$(\alpha\beta)^2(\alpha\gamma)^2(\beta\gamma)^2$	$(\alpha-\beta)^2(\alpha-\gamma)^2(\beta-\gamma)^2$	$\{HH'\}^2$	

TABLE II.

No.	Name.	Cayley Symbol.	A. C. Symbol.	Root Symbol.	Transvectant.	Remarks.
			QUARTIC.			
1*	H	$\overline{12}^2$	$(ab)^2a^2b^2$	$\Sigma(\alpha\beta)^2\gamma^2\delta$	$\{ff'\}^2$	
2*	$S(i)$	$\overline{12}^4$	$(ab)^4$	$\Sigma(\alpha\beta)^2(\gamma\delta)^2$	$\{ff'\}^4$	See Art. 46
3*	$J(T)$	$\overline{12}^2 . \overline{13}$	$(ab)^2(ac)ab^2c^3$	$\Sigma(\alpha\beta)^2(\alpha\gamma)\beta\gamma^2\delta^3$	$\{fH\}$	See Art. 48
4*	$T(j)$	$\overline{12}^2 . \overline{23}^2 . \overline{13}^2$	$(ab)^2(bc)^2(ac)^2$	$\Sigma(\alpha\beta)^2(\gamma\delta)^2(\alpha\gamma)(\beta\delta)$	$\{fH\}^4$	See Art. 47

* A unique form.

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